Optimum design of composite axi-symmetric pressure vessels domes

Navidfar M.H.¹ *, Gharehbaghi M.²
Process Development & Equipment Technology Department
Research Institute of Petroleum Engineering

Abstract

The aim of this investigation is to obtain an optimum convex shape for the dome of composite pressure vessels. The angle of ply orientations is taken as combinations of 0, +45, -45, and 90 degrees, with failure criterion of Tsi-Wu for each layer under the maximum design pressure. The efficiency of the vessel used as the objective for maximization is defined as the shape factor \( K = \frac{PV}{W(\sigma/\gamma)} \), with the parameters \( P, V, W, \sigma/\gamma \) as defined in the text [2]. The optimization method used is the non-gradient approach of the Complex method, and the generator curve of the dome is defined in terms of the radii to equidistant points along the axis of symmetry treated as the design variables. The design points along the generator are turned into a smooth curve by connecting them with cubic splines. The procedure shows remarkable success as exhibited by several examples worked out.

Key Words: Composite shells; Optimum curve; Domes; Minimum thickness; Pressure Vessels; Efficiency

1. Introduction

Fiber composite pressure vessels are widely used in the industry due to their high strength and low weight. In particular in the space technology they have extensive application as fuel or oxygen tanks or rocket combustion chambers. Most types such vessels are made of two main parts, i.e. a cylindrical main body and a dome shaped as dish ends. Normally the dome shaped ends are more susceptible to failure than the main body, and deserve the primary attention in design. Main consideration in the design of the dome ends is their geometric shape. Other considerations in their design include the material strength as well as the geometry of the filaments in the composite layers [1-15].

2. Dome Modeling

2.1. Shape Geometry

The axi-symmetric dome surface generator curve in the polar coordinates can be described as \( r = r(\theta) \). Main radii of the curve as shown in figure (1) are given in equations (1) and (2) [1].

\[
r_1 = -\frac{\left[1 + (r')^2\right]^{3/2}}{r''}
\]

\[
r_2 = r\left[1 + (r')^2\right]^{1/2}
\]
2.2. Resolution of the shell forces:

Following the membrane shell theory, the forces applied on a representative unit shell element are shown in figure (2).

The forces acting on the element shown in figure (2), can be derived from membrane axi-symmetric shell theory with axi-symmetric loading as equations (3) and (4), along the parallels or the meridians respectively [16-17].

\[ N_\phi 2\pi r \sin \phi = -\int_0^\phi 2\pi r (Yr_1 d\phi \sin \phi + Zr_1 d\phi \cos \phi) \]  
\[ N_\phi / r_1 + N_\theta / r_2 + z = 0 \]  

If \( Y = 0, \quad Z = -p \) is used in equation (3) and (4) the following reduced form results.

\[ N\phi = N_\phi = \frac{pr_2}{2} \]  
\[ N\theta = N_\theta = \frac{pr_2}{2} \left( 2 - \frac{r_1}{r_2} \right) \]  

For the case of axi-symmetric loading it can be shown that:

\[ N_{\phi\theta} = 0 \]
3. Composite laminated shells

Using the classical laminated composite theory [18], in the above defined axi-symmetric shell dome under uniform loading, stresses and strains in each layer in the material axes can be found and the failure possibility of the layers investigated. Due to the assumed symmetric stacking order of the layers in the shell thickness no coupling between the extensional and the bending forces and displacements can occur. Therefore, referring to figure (3) below, according to the classical lamination theory (CLT), the following relationships for the analysis of the laminate can be written.

![Figure 3: Representative laminated composite element.](image)

\[
\{N\}_{x0} = [A]\{e\}_{x0} \quad (8)
\]
\[
\{N\}_{x0} = [N_{\phi}, N_{\theta}, N_{\phi \theta}]^T \quad (9)
\]
\[
\{e\}_{x0} = [e_{\phi}, e_{\theta}, e_{\phi \theta}]^T \quad (10)
\]
\[
[A] = \sum_{k=1}^{n} [\bar{Q}]^k . (t_k) = t [\bar{Q}]^k \quad (11)
\]

In equation (11), \(t_k\) is the thickness of the k-th layer. The elements of the extensional stiffness matrix \([\bar{Q}]^k\) for an arbitrary angle of the fiber orientation are described in Appendix A. Stress strain relation for the kth layer for the \(\theta-\Phi\) geometric axes in figure (3) can be described as:

\[
\{\sigma\}_{x0} = [\bar{Q}]^k . [\bar{e}]_{x0} = [\bar{Q}]^k . \{e\}_{x0} \quad (12)
\]

Transformation of stresses from the material principle axes to the geometric axes is performed by:

\[
\{\sigma\}_{x0} = [T]^{-1} \{\sigma\}_{LT} \quad (13)
\]

where \([T]\) is the transformation matrix, described in Appendix A.

Using equation (8) and (13) to simplify equation (12), the following equation would result.

\[
\{\sigma\}_{LT} = [T][\bar{Q}]^k . [A]^T . \{N\}_{x0} \quad (14)
\]

Equation (14) can be rewritten as:

\[
\begin{bmatrix}
\sigma_{L}^{(k)} \\
\sigma_{T}^{(k)} \\
\sigma_{LT}^{(k)}
\end{bmatrix} =
\begin{bmatrix}
k_{11} & k_{21} & 0 \\
k_{21} & k_{22} & 0 \\
k_{31} & k_{32} & 0
\end{bmatrix}
\begin{bmatrix}
N_{\phi} \\
N_{\theta} \\
N_{\phi \theta}
\end{bmatrix} \quad (15)
\]
Elements of $K_{i,j}$, (I, j=1,2,3) are given in Appendix A.

4. Optimization Method

The optimization method chosen to maximize the efficiency parameter of the dome under the strength constrains the Complex algorithm of M.J. Box [19]. This non-gradient method has the ability to handle nonlinear programming problems with relatively low number of function evaluations or analysis steps required. The main flow chart of the algorithm is shown in Appendix B.

4.1. Objective function

An important parameter in the optimization of pressure vessels is the efficiency factor of the vessel. In this investigation the efficiency factor is defined as the shape factor defined as

$$K = \frac{PV}{W(\sigma/\gamma)},$$

in which $P, V, W, \sigma/\gamma$ are respectively the internal pressure, internal volume, dome weight, and the ratio of strength factor to the density. In case of the composite materials, the strength can be simplified to the tensile strength along the fibers. In order to simplify the mathematics involved, following the definition of the dome curvature by $r = r(z)$, the following non-dimensional parameters can be defined.

$$\rho \equiv r/r_c, \quad \zeta \equiv z/r_c, \quad \frac{d}{d\zeta} (\), \quad \vec{T} = \left( \frac{2X_t}{pr_c} \right) t$$

in which $X_t, r_c, t, p$ are the tensile strength along the fibers, the radius at the base of the dome, the shell thickness and the internal pressure. Thus the final form of the objective function using the above terms emerges as:

$$K = \frac{\int_0^{\xi_0} \rho^2 d\zeta}{\int_0^{\xi_0} \rho \sqrt{1 + (\rho^t)^2} d\zeta}$$

The objective is to maximize the above function subject to the limitations in the design.

4.2. Design variables

The dome generator curve is define by the coordinates of $n$ points on its surface which are equidistant along the vertical axis of symmetry; with radial coordinate $\rho$, and the axial coordinate $\zeta$ which are non-dimensionally defined as $\rho \equiv r/r_c$, $\zeta \equiv z/r_c$. $r_c$ is the radius at the base of the dome where it connects to the main body of the vessel. The coordinates of these points and the constant thickness of the shell are the design variables which are shown in figure (4).

To design a smooth curve for the dome shell passing thru the above points, cubic splines curves have been used. Since they constitute a different function between any two design points, no unique function can describe the dome’s generator curve. The use of a 6th degree polynomial approximation happens to adequately serve as a single function representation of the composite curve of the dome generator [20], as shown in the examples presented herein.
4.3. Design Constraints

Design constraints include the boundary conditions, e.g. the radius at the apex of the dome, and at the base, and the convexity requirement of the dome. Behavior constraints such as the strength requirements for the layers of the composite shell are also among the major design considerations. From among the numerous failure criteria such as maximum stress, strain and Tsai-Hill, the criterion of Tsai-Wu is chosen as the strength criterion to be used herein.

\[ F_1 \left( \sigma_L^{(k)} \right)^2 + F_2 \left( \sigma_T^{(k)} \right)^2 + F_6 \left( \sigma_{TL}^{(k)} \right)^2 + 2F_1F_2 \sigma_L^{(k)} \sigma_T^{(k)} + F_2 \sigma_L^{(k)} + F_2 \sigma_T^{(k)} - 1 \leq 0 \]

(18)

\( \sigma_L, \sigma_T, \sigma_{TL} \) are respectively the longitudinal, lateral and shear stresses of the layers in their principle material axes. The parameters in the assessment of failure are:

\[ F_1 = \frac{1}{\sigma_{LU}^r \sigma_{LU}^c}, F_2 = \frac{1}{\sigma_{LU}^l \sigma_{LU}^c}, F_6 = \frac{1}{\sigma_{TLU}^l} \]

\[ F_1 = \frac{1}{\sigma_{LU}^r - \sigma_{LU}^l}, F_2 = \frac{1}{\sigma_{LU}^l}, \]

\[ F_1 = \frac{1}{\sqrt{\sigma_{LU}^r \sigma_{LU}^c \sigma_{LU}^l}} \]

(19)

with \( \sigma_{LU}^r, \sigma_{LU}^l, \sigma_{TU}^r, \sigma_{TU}^l, \sigma_{TLU} \) being respectively the longitudinal tensile and compressive stresses and lateral tensile and compressive strengths.

Thus the complete problem of optimizing the dome shell for maximum efficiency factor under the above explained constraints is formulated and solved by the method of Complex. Details of the method of Complex is given in reference [19].

5. Examples

The example given here can serve to show the power and efficacy of the method developed in this investigation.
A dome of constant thickness with $r_0 = 0.2m, r_c = 0.5m, H = 0.5m$ made of Graphite/Epoxy of T300/5208 material with 70% fiber content is to be designed. 

Ply angles used are $0, -45, 45, 90$

Material properties data are:

$E_L = 181 GPa, E_T = 10.3 GPa, G_LT = 7.17 GPa,$

$v_{LT} = 0.28, \sigma_{LU} = 1500 MPa, \sigma_{TU} = 40 MPa,$

$\sigma_{TLU} = 68 MPa, \sigma_{TLU}' = 1500 MPa, \sigma_{TU}' = 40 MPa$

$E_L, E_T, \text{and } G_{LT}$ are the elastic modulus longitudinally, laterally and in shear, respectively. $v_{LT}$ is the Poisson’s ratio and $\sigma_{LU}, \sigma_{TU}, \sigma_{TLU}, \sigma_{TLU}'$ are the failure strengths as defined earlier. The dome is to operate under 200 bar pressure.

The solution curve is shown in figure (5), which also shows the continuous function describing the curve as a polynomial. Figure (6) and (7) respectively, show variations of the forces in the meridional and parallel directions in the height of the dome. Thicknesses for the optimum design are found for three alternate arrangements and are listed in Table 1.

![Figure 5: Generator curve for the example dome.](image-url)
Figure 6: Variation of force in the shell along the meridians in the example.

Figure 7: Variation of force perpendicular to the meridians in the example.
Table 1: Optimum thickness variations versus ply angle variations

<table>
<thead>
<tr>
<th>Stacking Sequence</th>
<th>0/90</th>
<th>0/−45/+45/90</th>
<th>−45/+45</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Sell Thickness</td>
<td>32mm</td>
<td>33mm</td>
<td>40mm</td>
</tr>
</tbody>
</table>

Figure 8: Generator curve for the example dome

\[ r = -476.25z^2 + 445.54z^3 - 170.71z^4 + 28.647z^5 - 2.3677z^6 + 0.0405z + 0.4395 \]
Figure 9: Variation of force in the shell along the meridians in the example

Figure 10: Variation of force perpendicular to the meridians in the example

Table 2: Optimum thickness variations versus ply angle variations

<table>
<thead>
<tr>
<th>Stacking Sequence</th>
<th>0/90</th>
<th>0/-45/+45/90</th>
<th>-45/+45</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Shell Thickness</td>
<td>36mm</td>
<td>37mm</td>
<td>43mm</td>
</tr>
</tbody>
</table>
Figure 11: Generator curve for the example dome

\[ t = -2717.5z^6 + 1950.1z^5 - 574.7z^4 + 74.796z^3 - 4.7935z^2 + 0.0717z + 0.4964 \]

Figure 12: Variation of force in the shell along the meridians in the example
Figure 13: Variation of force perpendicular to the meridians in the example

Table 3: Optimum thickness variations versus ply angle variations

<table>
<thead>
<tr>
<th>Stacking Sequence</th>
<th>0/90</th>
<th>0/−45/+45/90</th>
<th>−45/+45</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Sell Thickness</td>
<td>44mm</td>
<td>46mm</td>
<td>48mm</td>
</tr>
</tbody>
</table>

6. Conclusions

A method for the optimum design of composite pressure vessel domes with constant thickness and pre assigned ply angles and stacking order is developed herein. The method uses the axi-symmetric shell membrane theory and the analysis is performed for uniform internal pressure. The generator curve is optimized using a non-gradient approach via the Complex algorithm in terms of the coordinates of the design points. Cubic spline is used to make a smooth curve for the generator. A polynomial approximation is used to have a continuous function for the optimal generator curve. Objective function is the efficiency factor which is a function of the weight, volume, pressure, and strength.

An example is presented to show the success of the procedure, which can be effectively used in design of the end dome of pressure vessels in many branches of industry.

Appendix A

The elements of the reduced stiffness matrix $\bar{Q}$ of a laminate

$\bar{Q}_{11} = Q_{11}c^4 + 2(Q_{12} + 2Q_{66})c^2s^2 + Q_{22}s^4$

$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})c^2s^2 + Q_{12}(c^4 + s^4)$

$\bar{Q}_{22} = Q_{11}s^4 + 2(Q_{12} + 2Q_{66})c^2s^2 + Q_{22}c^4$

$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12})c^2s^2 + Q_{66}(c^2 - s^2)^2$

$\bar{Q}_{16} = -Q_{22}c^3s^3 + Q_{11}c^3s - (Q_{12} + 2Q_{66})(c^2 - s^2)cs$

$\bar{Q} = -Q_{22}c^3s + Q_{11}cs^3 - (Q_{12} + 2Q_{66})(c^2 - s^2)cs$
The elements of the standard transformation matrix $[T]$

$$
[T] = \begin{bmatrix}
c^2 & s^2 & 2cs \\
s^2 & c^2 & -2cs \\
-2cs & cs & (c^2 - s^2)
\end{bmatrix}
$$

Where $c = \cos \beta$, $s = \sin \beta$ is the winding angle of filament.

The elements of the lamina stiffness matrix $[Q]$

$$Q_{11} = \frac{E_L}{1 - \nu_{LT} \nu_{TL}}$$

$$Q_{22} = \frac{E_T}{1 - \nu_{LT} \nu_{TL}}$$

$$Q_{12} = \frac{\nu_{LT} E_T}{1 - \nu_{LT} \nu_{TL}}$$

$$Q_{16} = Q_{26} = 0$$

$$Q_{66} = G_{LT}$$

Elements of Matrix $K_{i,j}$:

The constants $k_{ij}(i,j = 1,2,3)$

$$k_{11} = c^2 + 2cs \frac{Q_{22} Q_{16} - Q_{12} Q_{26}}{Q_{11} Q_{22} - Q_{12}^2}$$

$$k_{12} = s^2 + 2cs \frac{Q_{11} Q_{26} - Q_{16} Q_{22}}{Q_{11} Q_{22} - Q_{12}^2}$$

$$k_{21} = s^2 - 2cs \frac{Q_{22} Q_{16} - Q_{12} Q_{26}}{Q_{11} Q_{22} - Q_{12}^2}$$

$$k_{22} = c^2 - 2cs \frac{Q_{11} Q_{26} - Q_{16} Q_{22}}{Q_{11} Q_{22} - Q_{12}^2}$$

$$k_{31} = -sc + (c^2 - s^2) \frac{Q_{22} Q_{16} - Q_{12} Q_{26}}{Q_{11} Q_{22} - Q_{12}^2}$$

$$k_{32} = sc + (c^2 - s^2) \frac{Q_{11} Q_{26} - Q_{16} Q_{22}}{Q_{11} Q_{22} - Q_{12}^2}$$
Appendix B:

Figure 14: Complex algorithm flow chart
References


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[19] Box MJ, A New method of constrained optimization and a comparison with other methods, Computer J, 8:42 (1965)